

Gravity from dynamical symmetry breaking

H.F. Westman^{1*} and T.G. Zlosnik^{2†}

¹*Instituto de Física Fundamental, CSIC,
Serrano 113-B, 28006 Madrid, Spain*

²*Imperial College Theoretical Physics,
Huxley Building, London, England*

(Dated: February 6, 2013)

It has been known for some time that General Relativity can be regarded as a Yang-Mills-type gauge theory in a symmetry broken phase. In this picture the gravity sector is described by an $SO(1, 4)$ or $SO(2, 3)$ gauge field $A^a_{b\mu}$ and Higgs field V^a which acts to break the symmetry down to that of the Lorentz group $SO(1, 3)$. This symmetry breaking mirrors that of electroweak theory. However, a notable difference is that while the Higgs field Φ of electroweak theory is taken as a genuine dynamical field satisfying a Klein-Gordon equation, the gauge independent component V^2 of the Higgs-type field V^a is typically regarded as non-dynamical. Instead, in many treatments V^a does not appear explicitly in the formalism or is required to satisfy $V^2 \equiv \eta_{ab}V^aV^b = \text{const.}$ by means of a Lagrangian constraint. As an alternative to this we propose a class of polynomial actions that treat both the gauge connection $A^a_{b\mu}$ and Higgs field V^a as genuine dynamical fields. The resultant equations of motion consist of a set of first-order partial differential equations. We show that for certain actions these equations may be cast in a second-order form, corresponding to a scalar-tensor model of gravity. A specific choice based on the symmetry group $SO(1, 4)$ yields a positive cosmological constant and an effective mass M of the gravitational Higgs field ensuring the constancy of V^2 at low energies and agreement with empirical data if M is sufficiently large. More general actions are discussed corresponding to variants of Chern-Simons modified gravity and scalar-Euler form gravity.

I. INTRODUCTION

The question of whether the gravitational field is essentially similar to other fields in nature or not has been a long running theme in physics. In particular, how far do the commonalities between gravitation and the gauge theories of particle physics extend? The basic ingredients of these gauge theories are gauge fields which are one-forms valued in the Lie algebra of some group G . In gauge theories with a spontaneously broken symmetry there are in addition to gauge fields Higgs fields which are scalar fields valued in some representation of G which may break the symmetry G down to a residual symmetry $H \subset G$ at the level of the equations of motion via the attainment of non-vanishing vacuum expectation values. An example of this in the standard model of particle physics is the electroweak theory. In that model the gauge fields are a $U(1)$ gauge field $B \equiv B_\mu dx^\mu$ coupling to hypercharge and an $SU(2)$ gauge field $C^A_B \equiv C^A_{B\mu} dx^\mu$ coupling to isospin, where A, B, \dots are $SU(2)$ indices. These fields are accompanied by a $U(1) \times SU(2)$ -valued field Φ^A , called the electroweak Higgs field. When $\Phi^\dagger \Phi \equiv \langle \Phi^\dagger \rangle \langle \Phi \rangle \neq 0$ the $SU(2) \times U(1)$ gauge symmetry is broken leaving a remnant $U(1)$ symmetry preserving the form of Φ^A . In this example then the group G is identified with $U(1) \times SU(2)$ and H is identified with the $U(1)$ symmetry of electromagnetism.

We now consider the gravitational field. This field is typically described entirely by a rank-2 space-time tensor $g_{\mu\nu}$ referred to as the metric tensor. We will refer to this as the second-order formulation of gravity. In the sense of the above definitions, this field is neither a gauge field nor a Higgs field [49]. However, the metric formulation of gravity may be indeed be recovered from ingredients that have more, but not everything, in common with gauge theory. This is possible via the first-order Einstein-Cartan formulation of gravity [1] wherein gravity is described by an $SO(1, 3)$ gauge field $\omega^I_J \equiv \omega^I_{J\mu} dx^\mu$, referred to as the spin-connection, and a Lorentz valued one-form $e^I \equiv e^I_\mu dx^\mu$, referred to as the co-tetrad. From the Einstein-Cartan perspective, gravitation is described by gauge invariant actions with the Lorentz group $SO(1, 3)$ as the local symmetry group G . The metric tensor is identified with $\eta_{IJ} e^I_\mu e^J_\nu$ where $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$ is the invariant $SO(1, 3)$ matrix.

While the connection ω^I_J has all the mathematical characteristics of a standard gauge field the co-tetrad e^I does not appear to have a counterpart in gauge theory [50]. It is a $SO(1, 3)$ -valued vector but also possesses a space-time index μ , a combination not found in any other known fundamental field in physics. This dissimilarity with gauge theory may ultimately be superficial if it turns out that the co-tetrad is a composite object formed of more familiar objects. For example, an approach which has attracted recent interest is to retain the $SO(1, 3)$ gauge field and regard e^I as just such a composite object. For instance [2–4] have considered models involving sets of $spin(1, 3)$ spinor fields $\{\Psi_{(i)}\}$ where

*Electronic address: westman@iff.csic.es

†Electronic address: t.zlosnik@imperial.ac.uk

$$e_\mu^I \equiv i \sum_{(i)} (\bar{\Psi}_{(i)} \gamma^I D_\mu \Psi_{(i)} - D_\mu \bar{\Psi}_{(i)} \gamma^I \Psi_{(i)}).$$

An alternative to the above spinorial approach is to regard the $SO(1,3)$ symmetry of the Einstein-Cartan theory as the remnant symmetry $H \subset G$ after spontaneous symmetry breaking, i.e. we begin with a symmetry group larger than $SO(1,3)$ and via symmetry breaking retain only local Lorentz symmetry. The ingredients necessary for this will thus be standard quantities from gauge theory: a gauge field $A_b^a \equiv A_{b\mu}^a dx^\mu$ and a Higgs field V^a where a, b, \dots are gauge indices of the larger group. The larger group can be taken to be the de Sitter $SO(1,4)$ group, the anti-de Sitter $SO(2,3)$, or the Poincare group $ISO(1,3)$. This approach is that of Cartan-geometry [5] where the gauge field and Higgs field admit a simple geometrical interpretation in terms of ‘idealized waywisers’ [6]. The gauge connection A_b^a dictates how much a symmetric space (either the de-Sitter, anti-de Sitter, or Minkowski spacetime) is rotated when rolling without slipping along some path on the spacetime manifold. The Higgs field V^a corresponds to the ‘point of contact’ between the symmetric space and the manifold.

If we are interested in eliminating differences between gravity and the gauge theories of particle physics, this approach is promising as one may use the above ingredients to construct an object which, like the co-tetrad, possesses a space-time index and a gauge index. This is simply the covariant derivative of V^a :

$$D_\mu V^a \equiv \partial_\mu V^a + A_{b\mu}^a V^b. \quad (1)$$

By construction, this quantity transforms as a one-form under coordinate transformations and as a vector under $SO(1,4)/SO(2,3)$ transformations. For concreteness we restrict attention to cases where the larger group is either of the 10-parameter groups $SO(1,4)$ and $SO(2,3)$. Consider a field V^a , where a, b, c, \dots now specifically refer to $SO(1,4)$ or $SO(2,3)$ indices. If $V^2 \equiv \eta_{ab} V^a V^b = -\ell^2$ for $SO(2,3)$ and $V^2 = \ell^2$ for $SO(1,4)$ then a gauge may be found where $V^a \stackrel{*}{=} \ell \delta_4^a$ (the notation $\stackrel{*}{=}$ denotes an equation holding in a particular preferred gauge), where ℓ is some real non-zero constant. One may check that the generators of $SO(1,4)/SO(2,3)$ which leave the above form of V^a invariant are those of the Lorentz group $SO(1,3)$. Therefore for the symmetry breaking ansatz $V^2 = \mp \ell^2$ we have $G = SO(1,4)/SO(2,3)$ and $H = SO(1,3)$. We will refer to any indices projected along V^a as 4 and the remaining $SO(1,3)$ indices as I, J, \dots . Given the ansatz for V^2 we have from (1) that:

$$D_\mu V^a \stackrel{*}{=} A_{4\mu}^a \ell \quad (2)$$

It can be seen that if $V^2 = \text{const.}$ then $D_\mu V^a$ only contains non-vanishing components orthogonal to V^a and transforms homogeneously under the $SO(1,3)$ transformations defined by V^a . Therefore $D_\mu V^a$ behaves like a co-tetrad. It additionally follows that A^I_J can play the role of the spin-connection ω^I_J . In these senses then, the familiar objects of Einstein-Cartan theory may be recovered in the symmetry broken phase of a

$SO(1,4)/SO(2,3)$ gauge theory. No objects beyond a gauge field and a Higgs field have been introduced.

Though this construction seems promising, it must be shown that $SO(1,4)/SO(2,3)$ invariant actions built from the pair $\{A_b^a, V^a\}$ exist that can reduce to Einstein-Cartan gravity, and thus General Relativity, after symmetry breaking. This was first shown to be possible by Stelle and West [7, 8] and later by Pagels [9]. Due to the relationship between the variables $\{A_b^a, V^a\}$ and Cartan geometry [10, 11], we refer to gravity seen through these variables as Cartan Gravity.

In these papers, the recovery of the co-tetrad via (2) was aided by actually enforcing the symmetry breaking ansatz $V^2 = \mp \ell^2$ by adding a Lagrange multiplier to the gravitational action. By analogy, in the electroweak theory this would be akin to fixing $\Phi^\dagger \Phi = \text{const.}$ via a similar constraint. It appears to be the case however that fluctuations of the scalar $\Phi^\dagger \Phi$, in the form of the electroweak Higgs particle, have been detected. Therefore in electroweak theory, variations in the action Φ^4 are not restricted. If indeed Cartan gravity represents a step closer to the remaining parts of physics, it would seem that fluctuations in $\eta_{ab} V^a V^b$ should similarly be allowed.

This implies that symmetry breaking solutions to a Cartan gravity model should follow from free variation of fields. We note that the authors of [7] indeed considered actions where V^a was allowed to vary. However, we note that these actions are non-polynomial in the fields $\{A_b^a, V^a\}$. As a simplifying principle we shall seek to consider only actions that are polynomial in these fields. Therefore our restrictions represent a point of departure from prior work [12–14].

The plan of this paper is as follows: In Section II we exhibit the most general polynomial Cartan gravity action. In Section III we discuss a choice of variables that enables the general action of II to be written in a particularly transparent form and facilitates reformulating the first-order theory in second-order form. In IV we consider a subset of this general action and show that these types of theories are equivalent to a class of scalar-tensor theories. As may be expected, the scalar degree of freedom is encoded in the norm V^2 . In Section V we consider the physical interpretation of more general actions, and in Section VI we discuss behaviour in the event that $V^2 = 0$. In Section VII we discuss the results of the previous sections.

Throughout the paper we work in units in which c and \hbar are dimensionless and numerically equal to one. Duration is measured in meters m and mass in inverse meters m^{-1} .

II. POLYNOMIAL ACTIONS FOR GRAVITY

In accordance with the discussion in the previous section we will look to construct gravitational actions from the pair $\{A_b^a, V^a\}$ which are each allowed to vary freely up to the restriction that $\delta A_b^a = \delta V^a = 0$ on

the boundary. As a further guide, we will additionally look to construct only actions that are invariant under $SO(1,4)/SO(2,3)$ and polynomial in the basic fields $\{A^a_b, V^a\}$. We shall see that these restrictions still allow a non-trivial collection of possible terms in the action and we shall here consider the most general combination of them.

Given our restrictions, what terms may be considered? In general these terms will be actions (defined as integrals over space-time four-forms) built from the invariants $\eta_{ab}, \epsilon_{abcde}$, the field V^a , and the curvature two-form of A^a_b defined as $F^a_b = \frac{1}{2}F^a_{b\mu\nu}dx^\mu \wedge dx^\nu$ where $F^a_{b\mu\nu} \equiv 2\partial_{[\mu}A^a_{b|\nu]} + 2A^a_{c[\mu}A^c_{b|\nu]}$. The $SO(1,4)/SO(2,3)$ indices may be raised with η^{ab} and lowered with η_{ab} . For notational compactness we denote the wedge product $y \wedge z$ between differential forms y and z simply as yz . For ex-

ample, if y is a three-form and z is a one-form then we have:

$$\begin{aligned} \int yz &= \int y \wedge z \\ &= \frac{1}{3!} \int y_\mu z_{\nu\sigma\delta} dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\delta \\ &= \frac{1}{3!} \int \varepsilon^{\mu\nu\sigma\delta} y_\mu z_{\nu\sigma\delta} d^4x \end{aligned}$$

where $\varepsilon^{\mu\nu\sigma\delta}$ is the Levi-Civita density. For an exposition of the calculus of variations using differential forms we refer to [6]. The most general action consistent with our requirements is as follows:

$$\begin{aligned} S[V^a, A^{ab}] &= \int (\alpha_1 \epsilon_{abcde} V^e + \alpha_2 V_a V_c \eta_{bd} + \alpha_3 \eta_{ac} \eta_{bd}) F^{ab} F^{cd} + (\beta_1 \epsilon_{abcde} V^e + \beta_2 V_a V_c \eta_{bd} + \beta_3 \eta_{ac} \eta_{bd}) DV^a DV^b F^{cd} \\ &\quad + \gamma_1 \epsilon_{abcde} V^e DV^a DV^b DV^c DV^d \end{aligned} \quad (3)$$

The α, β , and γ may themselves be polynomial functions of V^2 . The action (3) may look unfamiliar. It contains terms quadratic in the curvature F^{ab} but terms contributing to the equations of motion necessarily couple to the gravitational Higgs field V^a ; the only such term which may be independent of V^a is the α_3 term. However, if α_3 is constant then the term is simply proportional to $\int F_{ab} F^{ab} = \int d(A^{ab} F_{ab} + \frac{1}{3} A^{ac} A_a^d A_{cd})$ and so may be neglected as a boundary term.

III. DECOMPOSING THE CONNECTION

The equations of motion deduced from a variational principle of polynomial actions written in terms of forms and gauge covariant derivatives can be shown to invariably be first-order partial differential equations [15]. Thus, the above polynomial action (3) will lead to first-order partial differential equations in the dynamical variables V^a and A^a_b which do not look very familiar. In order to see how one may recover more familiar looking second-order differential equations we shall in this section introduce a ‘covariant’ decomposition of the connection A^a_b . We first recall a simplifying strategy from the Einstein-Cartan theory of gravity where gravity is described by the pair $\{\omega^I_J, e^I\}$. However, in doing variations of actions it is convenient to make the following decomposition:

$$\omega^{IJ} = \bar{\omega}^{IJ} + \mathcal{C}^{IJ} \quad (4)$$

where $\bar{\omega}^{IJ}$ is determined *entirely* by e^I and its partial derivatives via the following equation:

$$D^{(\bar{\omega})} e^I = 0. \quad (5)$$

which can be solved yielding

$$\bar{\omega}^{IJ}_\mu = 2(e^{-1})^{\nu[I} \partial_{[\mu} e^J_{\nu]} + e_{\mu K} (e^{-1})^{\nu I} (e^{-1})^{\alpha J} \partial_{[\alpha} e^K_{\nu]}$$

where $(e^{-1})^I_\mu$ is the inverse of the matrix e^I_μ referred to as the tetrad. By inspection $\bar{\omega}^{IJ}$ transforms as a gauge field and thus alone becomes responsible for the inhomogeneous transformation law of ω^{IJ} ; the one-form \mathcal{C}^{IJ} , referred to as the contorsion, transforms homogeneously under a gauge transformation [16, 17]. The two-form $D^{(\omega)} e^I$ is referred to as the torsion and hence, as a solution to (5), $\bar{\omega}^{IJ}$ is often referred to as the torsion-free spin-connection. The equations of motion for the Einstein-Cartan theory may be obtained by varying e^I and \mathcal{C}^{IJ} independently. A reason for why this is useful is that the equations of motion for \mathcal{C}^{IJ} take a particularly simple form in Einstein-Cartan theory in the presence of the matter fields of the standard model of particle physics. Indeed, of these fields it is only spinorial matter that is expected to couple to the contorsion [18]; the coupling is such that one can solve algebraically for the contorsion in terms of spinor current. This allows the contorsion to be reinserted into the action and eliminated from the variational principle. The resultant variational principle, dependent upon e^I and $\bar{\omega}^{IJ}(e^K)$, corresponds to the Einstein-Hilbert action and after the addition of the Gibbons-Hawking boundary term yields the Einstein equations for the metric tensor $g_{\mu\nu}$ following variation with respect to e^I .

We would like to make a similar decomposition in the Cartan gravity case. The idea again will be to encode the inhomogeneous transformation property of the connection in a quantity defined analogously to (5). The additional, homogeneously transforming terms in the connection will hopefully simplify analysis if some of them may be eliminated from the variational principle.

Consider the following decomposition of A^{ab} :

$$A^{ab} = \bar{A}^{ab} + B^{ab} \quad (6)$$

As for the case of $\bar{\omega}^{IJ}$ in the Einstein-Cartan model, the field \bar{A}^{ab} encodes the inhomogeneous transformation properties of the connection A^{ab} . We propose the following analogue of equation (5):

$$D^{(\bar{A})}D^{(A)}V^a = 0 \quad (7)$$

We will return to the interpretation of this equation shortly. The remaining part of the connection is the field B^{ab} , and by construction it transforms homogeneously with respect to $SO(1,4)/SO(2,3)$ gauge transformations. As follows, to simplify analysis we will assume that $V^a \neq 0$. From a dynamical perspective this restriction is not natural but we will see that the recovery of a four-dimensional metric in the theory is dependent the norm of V^a being non-zero. Given this assumption, we may define the quantity $E^a \equiv B^{ab}V_b$. Furthermore, we may define the field $C^{ab} \equiv B^{ab} - (2/V^2)E^{[a}V^{b]}$. We shall see that after symmetry breaking, E^a plays the role of the co-tetrad e^I whilst C^{ab} plays the role of the contorsion C^{IJ} . In this sense, the form B^{ab} unifies co-tetrad and contorsion into a single object. For the null case where $V^a \neq 0$, $V^2 = 0$ we shall see that though the field E^a is still well-defined, it corresponds to a metric tensor $\eta_{ab}E_\mu^a E_\nu^b$ which is three dimensional and therefore care is required.

In the event that $V^2 \neq 0$ it may be seen that (7) is satisfied if two independent conditions hold:

$$D^{(\bar{A})}V^a = \frac{1}{2V^2}dV^2V^a \quad (8)$$

$$D^{(\bar{A})}E^a = 0 \quad (9)$$

This implies the following solution:

$$\bar{A}^{ab} = -\frac{2}{V^2}dV^{[a}V^{b]} + \bar{\omega}^{ab} \quad (10)$$

where

$$\bar{\omega}^{ab}{}_\mu = 2(E^{-1})^{\nu[a}\partial_{[\mu}E_{\nu]}^{b]} + E_{\mu c}(E^{-1})^{\nu a}(E^{-1})^{\alpha b}\partial_{[\alpha}E_{\nu]}^c$$

Where we have used the field $(E^{-1})^{\mu a}$, the ‘inverse’ of E^a , which satisfies: $V_a(E^{-1})^{a\mu} = 0$, $E_\mu^a(E^{-1})^\mu_b = P^a{}_c P^c{}_b$, $E_{a\mu}(E^{-1})^{a\nu} = \delta_\mu^\nu$. The matrix $P^a{}_b \equiv \delta^a_b - V^a V_b/V^2$ acts to project along internal ‘directions’ orthogonal to V^a . By inspection the solutions for $\bar{\omega}^{IJ}$ and $\bar{\omega}^{ab}$ are identical up to interchange of $\{e_\mu^I, (e^{-1})^\mu_I\}$ with $\{E_\mu^a, (E^{-1})^\mu_a\}$. The notational coincidence is deliberate. As noted, after appropriate symmetry breaking induced by V^a the field E^a will play the roll of the co-tetrad e^I and $\bar{\omega}^{ab}$ will play the roll of the torsion-free spin-connection $\bar{\omega}^{IJ}$. We will assume that $\bar{\omega}^{ab}$ is well defined and so the applicability of this variable will be restricted by the conditions this places upon E^a . The additional V^a -dependent contribution to \bar{A}^{ab} encodes the inhomogeneously transforming part of $A^{ab}V_b$. By way of interpretation one may think of the five ‘degrees of freedom’ of V^a as being comprised of a norm V^2 and an orientation U^a unit-vector (i.e. $U^a \equiv V^a/\sqrt{V^2}$, $|\eta_{ab}U^a U^b| = 1$). If $V^2 \neq 0$ over a region of the manifold then one can choose a gauge where $U^a \stackrel{*}{=} \delta^a_4$. This fixes $\bar{A}_{ab} - \bar{\omega}_{ab} = 0$ implying that $A^{ab}V_b \stackrel{*}{=} E^a$.

In summary then, we adopt the following decomposition of A^{ab} :

$$A^{ab} = \bar{A}^{ab}(E^c, V^d) + C^{ab} + \frac{2}{V^2}E^{[a}V^{b]} \quad (11)$$

Consequently if $V^2 \neq 0$ and $\bar{\omega}^a_b$ is well defined we may consider the action (3) as a functional of $\{V^a, C^{ab}, E^a\}$ complemented by the constraints on the projections of $C_{ab}V^a = E_a V^a = 0$. Given the decomposition (11), we have

$$\begin{aligned} F^{ab} &= \bar{R}^{ab} - \frac{1}{V^2}E^a E^b + C^a{}_c C^{cb} + D^{(W)}C^{ab} \\ &\quad + \frac{2}{V^2}V^{[b}C^{a]}{}_c E^c + \frac{1}{V^4}E^{[a}V^{b]}dV^2 \\ DV^a &= \frac{1}{2V^2}dV^2V^a + E^a \end{aligned}$$

where $\bar{R}^{ab} = d\bar{A}^{ab} + \bar{A}^a{}_c \bar{A}^{cb}$. We can now look to write the individual contributions to the action $S[V^a, A^{ab}]$ in terms of the variables $\{V^a, C^{ab}, E^a\}$. It will prove useful to group the contributions in terms of the manner in which C^{ab} appears in the action. For some contributions terms with derivatives of C^{ab} can be eliminated by partial integration resulting in an action with C^{ab} entering only algebraically, whereas other terms are quadratic in derivatives of C^{ab} so that the derivatives $D^{(\bar{A})}C^{ab}$ cannot be removed by partial integration. The former contributions (after partial integration) are as follows:

$$\begin{aligned}
S_{\alpha_2}[V^a, E^a, C^{ab}] &= \int \left(\alpha_2 E^a E^b C_{am} C^m_b + \frac{\alpha_2}{V^2} dV^2 C_{ab} E^a E^b \right) \\
S_{\beta_1}[V^a, E^a, C^{ab}] &= \int \beta_1 \epsilon_{abcde} V^e E^a E^b \left(\bar{R}^{cd} - \frac{1}{V^2} E^c E^d + C^c_m C^{md} \right) \\
&\quad - \left(\frac{\partial \beta_1}{\partial V^2} + \frac{\beta_1}{2V^2} \right) \epsilon_{abcde} V^e dV^2 C^{ab} E^c E^d \\
S_{\beta_2}[V^a, E^a, C^{ab}] &= \int \frac{\beta_2}{2} dV^2 C_{ab} E^a E^b \\
S_{\beta_3}[V^a, E^a, C^{ab}] &= \int \left(\beta_3 E^a E^b C_{am} C^m_b + \frac{\beta_3}{V^2} \left(1 - \frac{V^2}{\beta_3} \frac{\partial \beta_3}{\partial V^2} \right) dV^2 C_{ab} E^a E^b \right) \\
S_{\gamma_1}[V^a, E^a] &= \int \gamma_1 \epsilon_{abcde} V^e E^a E^b E^c E^d
\end{aligned}$$

The remaining contributions, which involve derivatives of C^{ab} are:

$$\begin{aligned}
S_{\alpha_1}[V^a, E^a, C^{ab}] &= \int \alpha_1 \epsilon_{abcde} V^e \left(\bar{R}^{ab} \bar{R}^{cd} - \frac{2}{V^2} (\bar{R}^{ab} + C^a_m C^{mb}) E^c E^d + \frac{1}{V^4} E^a E^b E^c E^d \right) \\
&\quad - 2 \left(\frac{\partial \alpha_1}{\partial V^2} + \frac{\alpha_1}{2V^2} \right) \epsilon_{abcde} V^e dV^2 C^{ab} \left(\bar{R}^{cd} - \frac{1}{V^2} E^c E^d + \frac{1}{3} C^c_m C^{md} + \frac{1}{2} D^{(W)} C^{cd} \right) \\
S_{\alpha_3}[V^a, E^a, C^{ab}] &= \int \alpha_3 \bar{R}^{ab} \bar{R}_{ab} - 2 \frac{\partial \alpha_3}{\partial V^2} dV^2 C_{cd} \left(\bar{R}^{cd} - \frac{1}{V^2} E^c E^d + \frac{1}{3} C^c_m C^{md} + \frac{1}{2} D^{(W)} C^{cd} \right)
\end{aligned}$$

This distinction between the appearance of C^{ab} is important. For any combination of the actions for which C^{ab} appears algebraically (after a partial integration), the equation of motion for C^{ab} may be used to solve for C^{ab} itself. Consequently, this solution can be re-inserted back into the actions, yielding a variational principle that depends solely on V^a and E^a . In this case the contorsion field does not have independent degrees of freedom and is instead completely fixed by other fields. On the other hand, whenever the action contains contributions from the α_1 and α_3 terms the contorsion may not be solved for algebraically and thus should be regarded as a genuine dynamical field with independent degrees of freedom.

The above seven actions remain completely general up to the requirement that $V^2 \neq 0$ and that $\bar{\omega}^{ab}$ is well-defined. We will now concentrate on a sub-case which illustrates the consistency of the use of variables that assume $V^2 \neq 0$. Though the approach of using variables which require a non-zero V^2 is clearly restrictive we will see that the variables enable in a simple way the recovery of more familiar looking geometric objects such as $g_{\mu\nu}$ and the torsion-free Ricci scalar \mathcal{R} . Indeed, this may be seen as a generalisation of the relationship between Einstein-Cartan gravity and the second-order formulation of gravity. The second-order formulation follows

from the adoption of the ansatz (4) and does not admit more general solutions such as $e^I = 0, \omega^{IJ} \neq 0$ that vacuum Einstein-Cartan theory does. Clearly though, the description of gravity using the Einstein-Hilbert action and its variable $g_{\mu\nu}$ is justified if solutions to the equation of motion are consistent with its assumed invertibility, even if it is regarded as only sometimes recoverable from the Palatini action of Einstein-Cartan theory.

We postpone discussion of the possibility that $V^2 = 0$ to Section VI but we stress that from the perspective wherein V^a is a genuine dynamical field, *ad hoc* restrictions on V^2 (e.g. $V^2 > 0$, $V^2 \neq 0$, etc.) are rather unnatural. Instead, from the dynamical perspective we should in principle allow for possible solutions in which V^2 changes sign and can become zero on hyper surfaces. Such changes in the sign of V^2 will be accompanied with a change in metric signature.

Let us examine this a bit closer and also establish some notational conventions that will be used in the following sections. We will find that a metric formalism for gravity may be recovered and that the signature of the metric tensor will depend on the sign of V^2 for a given group. For instance for $SO(1, 4)$ if $V^2 > 0$ then the subgroup that leaves V^a invariant is $SO(1, 3)$ (corresponding to metric signature $(-, +, +, +)$). If $V^2 < 0$ then

the subgroup is $SO(4)$ (metric signature $(+, +, +, +)$). For $SO(2, 3)$ if $V^2 > 0$ then the subgroup is $SO(2, 2)$ (metric signature $(-, -, +, +)$) and for $V^2 < 0$ the subgroup is $SO(1, 3)$ (again corresponding to metric signature $(-, +, +, +)$). It is useful then to define a *positive definite* scalar field $\phi \equiv |\sqrt{V^2}|$ such that

$$\eta_{ab}V^aV^b = \sigma\phi^2 \quad (12)$$

The constant $\sigma = 1$ for $SO(1, 4)$ with $(-, +, +, +)$ metric signature and for $SO(2, 3)$ with $(-, -, +, +)$ signature, whilst $\sigma = -1$ for $SO(1, 4)$ with $(+, +, +, +)$ signature and for $SO(2, 3)$ with $(-, +, +, +)$ signature. Therefore in addition to keeping note of the norm of V^2 we must also keep track of the group being used as that will say what the metric signature is for a given V^2 . Therefore we introduce the symbol θ which takes the value -1 if the metric has an even number of timelike dimensions (inclusive of the case that there are 0) and the value $+1$ if the metric $g_{\mu\nu}$ has an odd number of timelike dimensions.

IV. A SPECIFIC EXAMPLE

We are now ready to work out the physics of specific choices of actions and cast it in a more familiar second-order form in terms of the metric tensor $g_{\mu\nu}$. Consider

then the case where, of the ' α, β, γ ', only β_1, β_2 are non-vanishing. Furthermore we will assume that these two quantities are constant i.e. do not depend on V^2 . As such β_1 and β_2 become constants with dimensions of m^{-3} and m^{-4} respectively. In the following V^a and E^a have dimensions m and the remaining objects such as $A^{ab}, C^{ab}, \bar{\mathcal{R}}^{ab}, \dots$ dimensionless.

The combined action is then:

$$S_1 = \int \beta_1 \epsilon_{abcde} V^e E^a E^b \left(\bar{R}^{cd} - \frac{1}{V^2} E^c E^d + C^c{}_m C^{md} \right) - \left(\frac{\beta_1}{2V^2} \epsilon_{abcde} V^e - \frac{\beta_2}{2} \eta_{ac} \eta_{bd} \right) dV^2 C^{ab} E^c E^d \quad (13)$$

If V^a is non-vanishing then it selects a subgroup which V^a is invariant under. If we again utilize the orientation unit-vector $U^a \equiv V^a / |\sqrt{V^2}|$ then $\epsilon_{abcd} \equiv \epsilon_{abcde} U^e$ is invariant under the remnant transformations. It can be seen from (13) that variation of the action with respect to C^{ab} will yield an equation linear in C^{ab} , which may be used to obtain the following solution for the field:

$$C_{ab} = \frac{1}{2V^2} E_{[a} \partial_{b]} V^2 + \frac{\beta_2 \theta}{8\beta_1 \phi} \epsilon_{abcd} \partial^c V^2 E^d$$

where $\partial_a V^2 \equiv (E^{-1})^\mu_a \partial_\mu V^2$. Substitution of C^{ab} back into (13) yields:

$$S_1[V^2, E^a] = \int \beta_1 \epsilon_{abcd} \phi E^a E^b \bar{R}^{cd} - \beta_1 \frac{1}{V^2} \epsilon_{abcd} \phi E^a E^b E^c E^d + \epsilon_{abcd} \frac{\beta_1 \phi}{16V^4} \left(1 - \frac{\beta_2^2 \theta \phi^2}{4\beta_1^2} \right) (6\partial_m V^2 \partial^b V^2 E^a E^m - \partial^m V^2 \partial_m V^2 E^a E^b) E^c E^d \quad (14)$$

We are now in a position to write S_1 in terms of more familiar variables. From E^a we may define a metric tensor $\bar{g}_{\mu\nu} \equiv \eta_{ab} E^a_\mu E^b_\nu$ and inverse metric $\bar{g}^{\mu\nu} \equiv \eta_{ab} (E^{-1})^{a\mu} (E^{-1})^{b\nu}$. This compels us to identify $\bar{R}_{\mu\nu\alpha\beta} = E^a_\mu E^b_\nu \bar{R}_{ab\alpha\beta}$ with the torsionless Riemann cur-

vature tensor. Furthermore we define the Ricci tensor $\bar{\mathcal{R}}^\mu{}_\nu \equiv \bar{R}^{\mu\alpha}{}_{\nu\alpha}$ and Ricci scalar $\bar{\mathcal{R}} \equiv \bar{\mathcal{R}}^\mu{}_\mu$. By standard methods, the action (14) may be rewritten as an integration over a scalar density:

$$S_1[\phi, \bar{g}_{\mu\nu}] = \int \beta_1 \left(2\bar{\mathcal{R}} + \frac{3}{\phi^2} \left(1 - \frac{\beta_2^2 \theta \phi^2}{4\beta_1^2} \right) \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{24\sigma}{\phi^2} \right) \phi \sqrt{-g} d^4x \quad (15)$$

We may cast the action in a more familiar form by considering the conformal transformation of the metric: $g_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu}$. The Ricci scalar \mathcal{R} corresponding to $g_{\mu\nu}$ is related to $\bar{\mathcal{R}}$ according to [19] $\bar{\mathcal{R}} = (\Omega^2 \mathcal{R} +$

$6g^{\mu\nu} \nabla_\mu \log \Omega \nabla_\nu \log \Omega + \bar{g}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \log \Omega$). If we choose $\Omega^2 = \phi/\phi_0$, for some constant ϕ_0 , the action in the new variables $(\phi, g_{\mu\nu})$ becomes

$$S_1[\phi, g_{\mu\nu}] = \int \left(2\beta_1\phi_0\mathcal{R} - \frac{3\phi_0\beta_2^2\theta}{4\beta_1}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 24\beta_1\sigma\frac{\phi_0^2}{\phi^3} \right) \sqrt{-g}d^4x \quad (16)$$

where the boundary term $\bar{\nabla}_\mu(6\beta_1\phi_0\bar{\nabla}^\mu\log\frac{\phi}{\phi_0})\sqrt{-g}$ has been removed. We can now identify the gravitational constant as $\kappa = 2\beta_1\phi_0 = \frac{1}{16\pi G}$ where G is Newton's constant. Furthermore, perhaps surprisingly, after this conformal transformation the 'wrong-signed' kinetic contribution of equation (15) is cancelled and the scalar field $\phi^2 = |V^2|$ is recognized as a scalar field satisfying a Klein-Gordon equation with potential $U(\phi) = 24\beta_1\sigma\frac{\phi_0^2}{\phi^3}$. This potential does not have a minimum and does not allow for a stable vacuum expectation value of ϕ . In such a case there is no privileged choice of the value of ϕ_0 which was introduced merely to make the conformal factor Ω dimensionless. However, if the potential could be modified by adding other polynomial terms to the action enabling a stable vacuum expectation value, then ϕ_0 could be identified with that vacuum expectation value. How to achieve this will be considered below in section IV C.

A. $SO(1,4)$ leading to $(-, +, +, +)$ metric signature

Recall that for the group $SO(1,4)$ then if $V^2 > 0$ then the invariant subgroup is $SO(1,3)$ and with our conventions on η_{ab} the metric $g_{\mu\nu}$ is of signature $(-, +, +, +)$. Therefore we have $\sigma = +1$ and $\theta = +1$. We can see from (16) that the action (13) corresponds to General Relativity coupled to a scalar field $\phi \equiv |\sqrt{V^2}|$ with right-sign kinetic term but with a potential with sign that depends upon the original gauge group. For $SO(1,4)$ we have a potential $1/\phi^3$. Therefore the field will tend to 'roll down' the potential to increasing values of ϕ .

B. $SO(2,3)$ leading to $(-, +, +, +)$ metric signature

For the group $SO(2,3)$ if $V^2 < 0$ then the invariant subgroup is $SO(1,3)$ and with our conventions on η_{ab} the metric $g_{\mu\nu}$ is of signature $(-, +, +, +)$, corresponding now to $\sigma = -1$ and $\theta = +1$. Now due to the different sign of we have the potential $-1/\phi^3$. This potential will tend to lead to runaway evolution toward $\phi = 0$. Therefore upon beginning from the action (13), we see that symmetry breaking is dynamically favoured for the case $SO(1,4)$.

C. Stabilizing ϕ

We have seen that the action (13) is a model of dynamical symmetry breaking in the $SO(1,4)$ case. However, the potential for the field ϕ does not have a minimum, and the model in the absence of matter corresponds to a Peebles-Ratra rolling quintessence model [20, 21]. We now consider additional polynomial contributions to the action that would alter the potential as to create local maxima and minima. By inspection, aside from the α_1 action it is only the γ_1 action that can provide additional contributions to the potential. All other terms involve the field C^{ab} which generically will depend upon derivatives of $|V^2| = \phi^2$. It is straightforward to rewrite the γ_1 action in terms of the variables of the previous section:

$$S_1[\phi, g_{\mu\nu}] = \int \left(2\beta_1\phi_0\mathcal{R} - \frac{3\phi_0\beta_2^2\theta}{4\beta_1}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi) \right) \sqrt{-g}d^4x \quad (17)$$

where we have defined the scalar field potential

$$U(\phi) \equiv 24\phi_0^2 \left(\frac{\beta_1\sigma}{\phi^3} - \frac{\gamma_1(\phi^2)}{\phi} \right). \quad (18)$$

A local minimum in the potential corresponds to a real, positive solution to the equations $(dU/d\phi)_{\phi=\phi_0} = 0$, $(d^2U/d\phi^2)_{\phi=\phi_0} > 0$.

Note that a peculiarity of this approach is that appears impossible to add a term to $U(\phi)$ corresponding to a cosmological constant (i.e. a constant part of $U(\phi)$). This would, for instance, correspond to a contribution to

γ_1 of the form $\sqrt{\sigma V^2}$ which is excluded by the requirement that the action (3) is polynomial in A^{ab} and V^a . However, if ϕ settles to a stationary point ϕ_0 then it creates an effective cosmological constant. Furthermore the quadratic term in an expansion around ϕ_0 defines a mass M of the scalar field.

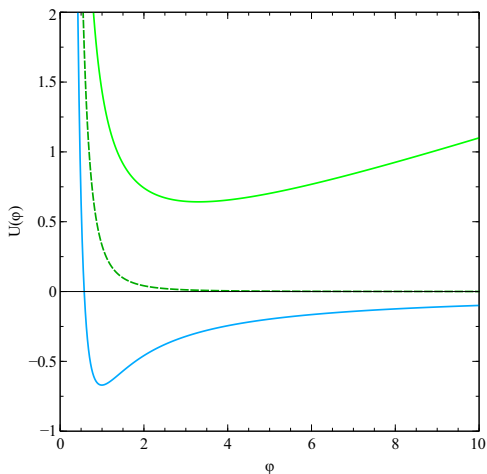


FIG. 1: Some functional forms of $U(\phi)$ for the $SO(1,4)$ case (units arbitrary). The dashed green line represents the $+1/\phi^3$ 'curvature' contribution to the potential. The blue curve shows the recovery of a stable global minimum (negative cosmological constant if at this point) for an additional $\sim -1/\phi$ due to constant γ_1 contribution whilst the light green curve shows the recovery of a stable global minimum (positive cosmological constant if at this point) for an additional $\gamma_1 \sim \text{const.} + \phi^2$.

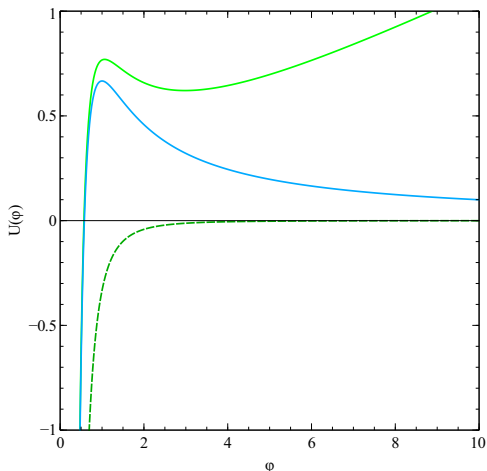


FIG. 2: Some functional forms of $U(\phi)$ for the $SO(2,3)$ case. The dashed green line represents the $-1/\phi^3$ 'curvature' contribution to the potential. The blue curve shows the recovery of an unstable maximum (positive cosmological constant if at this point) for an additional constant γ_1 contribution whilst the light green curve shows the recovery of a stable local minimum (positive cosmological constant) for an additional $\gamma_1 \sim \text{const.} + \phi^2$.

D. An empirically viable model

It is now interesting to examine whether there exist a specific choice of action which would lead to a modification of General Relativity but yet not ruled out by empirical fact. In particular, such model should imply a

stable positive cosmological constant at sufficiently low energies. Of the six specific models displayed in Fig. 1 and 2 only one model seems free of potential or immediate problems.

Clearly, the model corresponding to the blue line of Fig. 1 is ruled out because of its stable negative value of the cosmological constant emulated by $U(\phi_{min})$. We also note that all models based on $SO(2,3)$ seem potentially problematic as none of the potentials in Fig. 2 are bounded from below. This can potentially cause instability problems, especially in a quantum context with the possibility of quantum tunnelling. For this reason we will put them aside and regard them in absence of further analysis as unsuitable. Furthermore, the dashed green line in Fig. 1 corresponds to an asymptotically zero cosmological constant ($U(\phi) \rightarrow 0$ as $\phi \rightarrow \infty$) and it is therefore not clear without further analysis that this model would be empirically viable.

This leaves us with the model corresponding to the bright green line in Fig. 1 which is characterized by having a global stable minimum of the potential $U(\phi)$. This model is based on $SO(1,4)$, a spacelike contact vector V^a , with β_1 and β_2 as constants, and with $\gamma = -CV^2 = -C\phi^2$. We shall now see how the parameters β_1, β_2 , and C are related to more familiar constants such as the gravitational and cosmological constants κ and Λ and an effective mass M for the gravitational Higgs field V^a .

The model is then defined by the action (17) with the scalar field potential

$$U(\phi) = 24\phi_0^2 \left(\frac{\beta_1}{\phi^3} + C\phi \right) \quad (19)$$

corresponding to a choice of $\gamma(\phi^2) = -C\phi^2$ for some constant C of dimensions m^{-7} .

As mentioned above, the constant ϕ_0 was introduced to make the conformal factor $\Omega = \frac{\phi}{\phi_0}$ dimensionless. As such it is arbitrary. However, it is natural to fix it by requiring it to coincide with the value of ϕ at the potential minimum, i.e. we impose

$$0 = \left. \frac{\partial U}{\partial \phi} \right|_{\phi=\phi_0} = 24\phi_0^2 \left(-\frac{3\beta_1}{\phi_0^4} + C \right) \quad (20)$$

which implies

$$\phi_0^4 = \frac{3\beta_1}{C}. \quad (21)$$

Furthermore, whenever the scalar field has settled to its vacuum expectation value ϕ_0 it is constant everywhere and the corresponding value of the potential $U(\phi_0)$ will then emulate a cosmological constant Λ defined by $\int \kappa(\mathcal{R} - 2\Lambda)\sqrt{-g}$. The gravitational constant must be identified as $\kappa = 2\beta_1\phi_0$ and together with equation (21) we obtain $\Lambda = \frac{U(\phi_0)}{2\kappa} = \frac{48\beta_1}{\kappa\phi_0} = \frac{24}{\phi_0^2}$. We can now express β_1 and C in terms of Λ and κ

$$\beta_1 = \frac{\kappa\sqrt{\Lambda}}{\sqrt{96}} \quad C = \frac{\kappa\Lambda^{5/2}}{768\sqrt{6}}. \quad (22)$$

Thus, if the constants β_1 and C are chosen properly we reproduce the predictions of General Relativity plus a cosmological constant whenever the scalar field is in its ground state. Note that in the absence of matter, a ‘genuine’ cosmological constant truly independent of ϕ cannot be constructed- such a term could only arise from non-polynomial terms in the action (3) such as a contribution to γ_1 of the form $\sqrt{V^2}$.

At sufficiently high energies scalar field V^2 may get excited from its lowest energy state, the vacuum expectation value ϕ_0 . At which energy that happens is dictated by the effective mass M of the scalar field.

In order to determine the effective mass we need to put the scalar field kinetic term in a standard form. This is achieved by a rescaling $\phi = a\varphi$ with $a^{-2} = \frac{3\phi_0\beta_2^2}{4\beta_1}$. The action now takes on the standard form

$$\bar{S}_1[\varphi, g_{\mu\nu}] = \int (\kappa\mathcal{R} - g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - U(\varphi))\sqrt{-g}d^4x$$

with $U(\varphi) = 24\phi_0^2(\frac{\beta_1}{a^3\varphi^3} + C a\varphi)$. The effective mass M of the scalar field φ is defined as ($\phi_0 \equiv a\varphi_0$):

$$M^2 \equiv \frac{1}{2} \frac{\partial^2 U}{\partial \varphi^2} \Big|_{\varphi=\varphi_0} = \frac{1}{2} 24\phi_0^2 \frac{12\beta_1}{a^3\varphi_0^5} = \frac{\kappa^2 \Lambda^3}{288\beta_2^2} \quad (23)$$

and we see that for given values of κ and Λ it is the parameter β_2 that determines the effective mass of the scalar field φ .

We have now seen how General Relativity is recovered in the low energy limit from a gauge theory with dynamical symmetry breaking defined by the action

$$S[A_{b\mu}^a, V^a] = \int (\beta_1 \epsilon_{abcde} V^e + \beta_2 V_a V_c \eta_{bd}) DV^a DV^b F^{cd} - CV^2 \epsilon_{abcde} V^e DV^a DV^b DV^c DV^d.$$

containing the two basic ingredients of such a theory: a $SO(1,4)$ -valued gauge field $A_{b\mu}^a$ and a Higgs field V^a with norm V^2 subject to a Klein-Gordon equation. In particular, at the fundamental level neither metric nor co-tetrad are necessary for the formulation of this theory.

V. MORE GENERAL CASES

It is natural to ask how general the results of the previous section are. In Appendix A an identical calculation is performed for the case where all α, β, γ except α_1 and α_3 are non-zero and allowed to have a polynomial dependence upon V^2 . As for the case considered in Section IV C, if $V^2 \neq 0$ then the action is equivalent to a metric theory of gravity coupled to a scalar field. By inspection one may again make a conformal transformation of the metric to recover the familiar Einstein-Hilbert Lagrangian. To recover a canonical kinetic term for the scalar field, it is necessary to perform a redefinition of the scalar field.

Finally we consider the α_1 and α_3 terms. As mentioned, variation of these actions with respect to C^{ab} will yield field equations containing derivatives of C^{ab} . Therefore the presence of these terms in an action generally precludes the possibility of eliminating C^{ab} from the variational principle. In this sense these actions involve ‘propagating contorsion’. We now consider the impact of these terms.

A. Effect of the α_1 term

We first consider the limiting case where V^2 is constant and non-vanishing. The action in this limit is commonly referred to as the Macdowell-Mansouri action [22]. Here it may be checked that the term $\epsilon_{abcde} V^e \bar{R}^{ab} \bar{R}^{cd}$ reduces to a boundary term whereas the remaining terms (i.e. those that do not vanish when $dV^2 = 0$) amount to Einstein-Cartan gravity in the presence of a cosmological constant. The Macdowell-Mansouri action has been studied in a wide variety of contexts [11, 23–29]. When $V^2 \neq 0$ and varying, a number of terms in S_{α_1} that are proportional to dV^2 may now contribute. This action has been considered briefly in [30] but this model, with the remnant $SO(1,3)$ symmetry of the symmetry broken phrase, seems largely unexplored. Cosmological solutions for a very closely related action to S_{α_1} have been recently considered [31].

B. Effect of the α_3 term

The structure of the α_3 term is rather more familiar. Recall that in the event that V^2 is constant then $\alpha_3(V^2)$ is necessarily constant and the term is simply a boundary term. If $V^2 \neq 0$ and varying and α_3 carries a dependence on V^2 then the α_3 term corresponds to an additional term in gravitation in the Chern-Simons modified gravity theory [32–38], with α_3 playing the role of the Chern-Simons scalar field Φ_{cs} . Typically also present in these theories are an Einstein-Hilbert (or Palatini) action as well as an action for the dynamics of Φ_{cs} . A direct comparison between Chern-Simons modified gravity a subset of the general actions (3) requires some care. We have seen that the β_1 action may reduce to the Einstein-Hilbert action only after conformal transformation of E^a , therefore a more accurate comparison follows after an additional conformal transformation of the α_3 action. A persistent commonality though is the presence of derivatives of the contorsion, if allowed for, in the action of the theory. As in the case of the α_1 term, the presence of a non-vanishing $\alpha_3(V^2)$ may preclude the existence of a second-order formulation in terms of a metric and scalar field.

VI. NULL OR VANISHING V^a

It has become apparent in the previous sections that the tensor $\bar{g}_{\mu\nu} = \eta_{ac} V_b V_d B^a{}_\mu{}^b B^c{}_\nu{}^d = E^a{}_\mu E_{a\nu}$ is to be identified, up to a conformal factor, with the familiar metric tensor in situations where $V^a \neq 0$. However, dynamically it may be the case that V^a becomes null (i.e. $\eta_{ab} V^a V^b = 0$). This condition may be satisfied with $V^a = 0$ or $V^a \neq 0$. When $V^a = 0$ then $\bar{g}_{\mu\nu} = 0$ and so it is unclear whether a metric description exists in regions where this is satisfied. The null case with $V^a \neq 0$ is a bit more subtle. In this case we may choose a gauge where $V^a = \psi(x^\mu)(1, 0, 0, \pm 1)$, where first and last components refer to internal time and spatial components. The remaining components will be labelled by lowercase middle-alphabet Latin indices i, j, \dots

$$\bar{g}_{\mu\nu} = \psi^2 \eta_{ij} (B^i{}_0 \pm B^i{}_4)_\mu (B^j{}_0 \pm B^j{}_4)_\nu \quad (24)$$

Therefore in the null case the signature of the metric will be that of η_{ij} i.e. the signature will be three dimensional rather than four dimensional. For $SO(1, 4)$ then the metric has signature $(+, +, +)$ and for $SO(2, 3)$ the metric will have signature $(-, +, +)$. The α_1 action in isolation was studied for the former possibility in [39], wherein it was indeed found that a three dimensional geometric description was appropriate. Recall that for $SO(1, 4)$ when $V^2 > 0$ then the metric is of signature $(-, +, +, +)$ whereas when $V^2 < 0$ the metric is of signature $(+, +, +, +)$. It is tempting to wonder whether situations may occur where the transition of V^2 through 0 dynamically facilitates a change of metric signature. Such a possibility has been examined in the $(3+1)$ formulation of General Relativity [40]. Consideration of explicit models in Cartan Gravity, however, is beyond the scope of this current work.

VII. DISCUSSION AND OUTLOOK

A common view in physics has been that gravity should ultimately be thought of just another force field on par with the strong and electroweak fields of the standard model. From a mathematical perspective such a view is limited by the difference in mathematical machinery used by the force fields of particle physics on the one hand and the gravitational field on the other. More specifically, while the force fields of particle physics are mathematically represented by gauge connections valued in some suitable Lie algebra, the gravitational field is typically described by a symmetric second rank metric tensor $g_{\mu\nu}$. The mathematical difference is still present within the Einstein-Cartan formulation of gravity in terms of a $SO(1, 3)$ -valued gauge connection ω^{IJ} and a co-tetrad e^I . In particular, the co-tetrad regarded as a fundamental object appears to have no analogue within ordinary gauge theory [51].

However, it has been known for some time that gravity can be formulated exclusively in terms of objects of gauge theories with spontaneously broken symmetry. Within this Cartan-geometric formulation of gravity the descriptor of gravity consists of the pair $\{A^a{}_b, V^a\}$. These objects admit a simple geometrical interpretation in terms of ‘idealized-waywisers’ [6], but from a particle physics point of view we recognize $A^a{}_{b\mu}$ as a standard gauge connection and V^a as a symmetry breaking Higgs field. In this paper we have shown that there exist actions which are polynomial in these fields and their exterior derivatives which allow for a dynamical symmetry breaking mechanism closely mirroring that of the electroweak theory of particle physics. It is noteworthy that this is achieved without any metric tensor or co-tetrad appearing in the action. Instead, these objects are regarded as compound objects which can be constructed from the more fundamental variables $\{A^a{}_b, V^a\}$ should it be convenient to do so. In relation to this, we note that the constrained-norm form of Cartan gravity has been recovered from a perspective where V^A itself is a composite object, regarded as a fermionic condensate constructed from either $spin(1, 4)$ or $spin(2, 3)$ valued spinor fields [41].

To our knowledge the recovery of such symmetry breaking solutions following free variation of $\{A^a{}_b, V^a\}$ in a polynomial action principle is new. By way of contrast, the constancy of V^2 is typically imposed by hand within Cartan gravity without any dynamical mechanism proposed. From a pure mathematical point of view we can of course regard V^a as a mathematical redundant representation of a preferred section on the gauge fibre bundle that is necessary for the mathematical construction of Cartan geometry [42]. On that view the norm V^2 does not play any role in the construction and we might as well impose $V^2 = \text{const.}$. However, from a particle physics point of view, and in particular with the electroweak Higgs field in mind, the restriction $V^2 = \text{const.}$ does not appear natural. Instead, it suggests a natural modification of Cartan gravity in which the Higgs field V^a is treated as a genuine dynamical field subject to equations of motion. This was the basic motivation for this paper.

Although the class of actions (3), which always lead to first-order partial differential equations, look rather unfamiliar they may (excluding α_1 - and α_3 -terms) be recast into a more familiar second-order form where the actions reduce to particular examples of scalar-tensor theories. Furthermore, we showed how a specific subset of (3) lead to a scalar potential with a global minimum playing the role of a vacuum expectation value. Agreement with General Relativity is guaranteed as long as the effective mass M calculated from the potential is large enough to suppress the excitation of the Higgs field. Interestingly the requirement that (3) is polynomial in basic variables precludes the possibility of a genuine ‘bare’ cosmological constant term according to the volume form $\sqrt{-g} d^4x$. For high energies deviations from General Rel-

activity are expected due to the presence of gravitational Higgs bosons. A particularly stringent constraint on the mass of M may come due to the coupling of V^a to matter fields [15].

One perhaps surprising result of this paper is that $\phi \equiv |\sqrt{V^2}|$ appears as a standard scalar field when the action is rewritten in a second-order form. From a differential forms point of view this shows that the Hodge dual pops up naturally. The natural appearance of the Hodge dual when rewriting polynomial first-order field equations in a second-order form was also noted in [15]. This seems to indicate that the use of auxiliary fields, see e.g. [43, 44], may not be necessary in order to reproduce equations with such a Hodge structure, e.g. the Klein-Gordon and Yang-Mills equations.

One central simplifying assumption in the analysis of this paper was that V^2 be everywhere non-zero. We have argued that this is akin to the use of an invertible metric $g_{\mu\nu}$ in the variational principle of second-order General Relativity, the use of which is consistent whenever solutions to the resultant equations of motion do not threaten the assumption of invertibility. However, if we take the dynamical perspective seriously such a restriction on V^2 must be regarded as *ad hoc*. Instead, one should take the view that the field equations should dictate what possible solutions of V^a we can have. As such we cannot exclude the possibility of V^2 changing sign or of V^a even becoming zero on hypersurfaces. Under such conditions the scalar-tensor formalism breaks down but perhaps not the applicability of the basic variables $\{A^a_b, V^a\}$. As an interesting dynamical possibility we might have metric signature change. This would occur whenever V^2 changes sign, selecting out a different subgroup of $SO(1,4)/SO(2,3)$. Within a model based on $SO(1,4)$, V^2 may be spacelike in some region leading to the usual Lorentzian signature $(-, +, +, +)$ of the metric (based on subgroup $SO(1,3)$). However, it might be possible that the first-order field equations allow solutions where the sign of V^2 varies from region to region. In particular, in regions where V^2 is timelike, the ‘spacetime’ becomes Euclidean with signature $(+, +, +, +)$ (based on subgroup $SO(4)$). It would therefore be interesting to investigate whether there are solutions to the first-order field equations that admit a signature change and thus a dynamical and classical realization of Hartle and Hawking’s no-boundary proposal [45]. Indeed this represents a subtlety in this approach to viewing gravity as a gauge theory: changes in the character of the symmetry breaking or even symmetry restoration inevitably coincide with the loss of familiar notions of space and time.

The present work may be taken to suggest a more radical view on the form of field equations in Nature. Typically second-order field equations are regarded as the ones chosen in Nature. In particular, gravity, Yang-Mills fields, and scalar fields are all described by second-order partial differential equations. However, a notable exception are fermionic fields described by Dirac equations which are naturally on first-order. Further, the

Cartan-geometric formulation of gravity and the coupling to matter fields through the gauge prescription [15] puts all fields equations on first-order form. Of course, this may be regarded as a mere reformulation. However, as we have noted in this paper, the inclusion of terms like α_1 and α_3 will not allow for a second-order equivalent. With such contributions to the action the contorsion field becomes a genuinely dynamical field which cannot be solved for algebraically in terms of other fields. However, whenever the scalar V^2 becomes constant the contorsion field can be solved for algebraically and a second-order formulation becomes possible. From this perspective, we may perhaps regard the ubiquitousness of second-order field equations in nature as a contingent feature of a universe where the Higgs field V^a has settled to its vacuum expectation value. However, at a more fundamental level the theory would be governed by first-order equations that cannot be cast into second-order ones.

As a further departure from the standard picture of gravity, it is conceivable that the $SO(1,4)/SO(2,3)$ invariance is itself part of a symmetry broken phase of a model with a larger symmetry. For instance, it may be interesting to construct an equivalent of the actions (3) for a gauge theory of the conformal group $SO(2,4)$; conceivably the dynamics of symmetry breaking may allow for a General Relativity limit, and more comprehensive differences at higher energies.

Acknowledgments

We thank Max Bañados, Brian Dolan, Pedro Ferreira, João Magueijo, and Adolfo Toloza for insightful discussions. HW was supported by the CSIC JAE-DOC 2011 program. TZ was supported by STFC grant ST/J000353/1

Appendix A: General Case

It may be shown after rather lengthy calculation that the case where only α_1 and α_3 are assuming to be zero and all other variables are allowed to depend polynomially upon V^2 leads to the following second-order action upon elimination of C^{ab} :

$$S_2[\phi, \bar{g}_{\mu\nu}] = \int \left(2\beta_1 \bar{\mathcal{R}} - \xi(\phi) \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{24(\sigma\beta_1 - \gamma_1\phi^2)}{\phi^2} \right) \phi \sqrt{-\bar{g}} d^4x$$

where

$$\xi(\phi) = 3 \frac{\left(\theta Q^2 - 4 \left(\frac{\partial(\beta_1 \phi)}{\partial \phi^2} \right)^2 + \frac{2\sigma(\alpha_2 + \beta_3)\theta}{\beta_1 \phi} \left(\frac{\partial(\beta_1 \phi)}{\partial \phi^2} \right) Q \right)}{\beta_1 \left(1 + \frac{\theta}{\phi^2} \left(\frac{\alpha_2 + \beta_3}{2\beta_1} \right)^2 \right)}$$

$$Q \equiv \frac{\sigma \alpha_2}{\phi^2} + \frac{\beta_2}{2} + \frac{\sigma \beta_3}{\phi^2} \left(1 - \frac{\phi^2}{\beta_3} \frac{\partial \beta_3}{\partial \phi^2} \right)$$

As before we may make a conformal transformation. If β_1 itself has a dependence upon V^2 we define we make the following conformal transformation $\bar{g}_{\mu\nu} = (\kappa/2\beta_1\phi)g_{\mu\nu}$, in terms of which the action becomes:

$$S_2[\phi, g_{\mu\nu}] = \int \left(\kappa \mathcal{R} - \left(\frac{\kappa}{2\beta_1} \right) \left(\frac{12}{\beta_1} \left(\frac{\partial(\beta_1 \phi)}{\partial \phi^2} \right)^2 + \xi(\phi) \right) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 24 \left(\frac{\kappa}{2\beta_1} \right)^2 \left(\frac{\sigma \beta_1}{\phi^3} - \frac{\gamma_1}{\phi} \right) \right) \sqrt{-g} d^4 x$$

where recall that the $\alpha_i, \beta_i, \gamma_i$ may depend polynomially on ϕ^2 and the constant $\kappa \equiv 1/16\pi G$.

-
- [1] T. Kibble, J.Math.Phys. **2**, 212 (1961).
 - [2] K. Akama, Prog.Theor.Phys. **60**, 1900 (1978).
 - [3] D. Diakonov (2011), 1109.0091.
 - [4] Y. N. Obukhov and F. W. Hehl, Phys.Lett. **B713**, 321 (2012), 1202.6045.
 - [5] D. K. Wise, SIGMA **5**, 080 (2009), 0904.1738.
 - [6] H. Westman and T. Zlosnik (2012), 1203.5709.
 - [7] K. S. Stelle and P. C. West, J. Phys. **A12**, L205 (1979).
 - [8] K. Stelle and P. C. West, Phys.Rev. **D21**, 1466 (1980).
 - [9] H. R. Pagels, Phys. Rev. **D29**, 1690 (1984).
 - [10] R. Sharpe (1997), book, Springer.
 - [11] D. K. Wise, Class.Quant.Grav. **27**, 155010 (2010), gr-qc/0611154.
 - [12] F. Wilczek, Phys.Rev.Lett. **80**, 4851 (1998), hep-th/9801184.
 - [13] M. Leclerc, Annals Phys. **321**, 708 (2006), gr-qc/0502005.
 - [14] R. Tresguerres, Int.J.Geom.Meth.Mod.Phys. **5**, 171 (2008), 0804.1129.
 - [15] H. Westman and T. Zlosnik (2012), 1209.5358.
 - [16] M. Leclerc, Int.J.Mod.Phys. **D16**, 655 (2007), gr-qc/0506080.
 - [17] F. W. Hehl and Y. N. Obukhov (2007), 0711.1535.
 - [18] F. Hehl and B. Datta, J.Math.Phys. **12**, 1334 (1971).
 - [19] R. M. Wald (1984), book, The University of Chicago Press.
 - [20] B. Ratra and P. Peebles, Phys.Rev. **D37**, 3406 (1988).
 - [21] P. Peebles and B. Ratra, Astrophys.J. **325**, L17 (1988).
 - [22] S. W. MacDowell and F. Mansouri, Phys. Rev. Lett. **38**, 739 (1977), [Erratum-ibid.38:1376,1977].
 - [23] D. K. Wise, J.Phys.Conf.Ser. **360**, 012017 (2012), 1112.2390.
 - [24] G. W. Gibbons and S. Gielen, Class.Quant.Grav. **26**, 135005 (2009), 0902.2001.
 - [25] A. Randono (2010), 1010.5822.
 - [26] S. Gryb and F. Mercati (2012), 1209.4858.
 - [27] O. Obregon, M. Ortega-Cruz, and M. Sabido, Phys.Rev. **D85**, 124061 (2012).
 - [28] R. Durka (2012), 1208.5185.
 - [29] L. Freidel and S. Speziale, SIGMA **8**, 032 (2012), 1201.4247.
 - [30] M. Banados, Phys.Rev. **D55**, 2051 (1997), gr-qc/9603029.
 - [31] A. Toloza and J. Zanelli (2013), 1301.0821.
 - [32] S. Alexander and N. Yunes, Phys.Rev. **D77**, 124040 (2008), 0804.1797.
 - [33] M. B. Cantcheff, Phys.Rev. **D78**, 025002 (2008), 0801.0067.
 - [34] U. Ertem (2012), 0912.1433.
 - [35] S. Alexander and N. Yunes, Phys.Rept. **480**, 1 (2009), 0907.2562.
 - [36] D. Garfinkle, F. Pretorius, and N. Yunes, Phys.Rev. **D82**, 041501 (2010), 1007.2429.
 - [37] C. Furtado, J. Nascimento, A. Y. Petrov, and A. Santos (2010), 1005.1911.
 - [38] P. Canizares, J. Gair, and C. Sopuerta, J.Phys.Conf.Ser. **363**, 012019 (2012), 1206.0322.
 - [39] H. Westman and T. Zlosnik (2012), 1201.2725.
 - [40] G. Ellis, A. Sumeruk, D. Coule, and C. Hellaby, Class.Quant.Grav. **9**, 1535 (1992).
 - [41] A. Randono, Class. Quant. Grav. **27**, 215019 (2010), 1005.1294.
 - [42] R. J. Petti, Class. Quant. Grav. **23**, 737 (2006).
 - [43] L. Smolin, Phys.Rev. **D80**, 124017 (2009), 0712.0977.
 - [44] A. G. Lisi, L. Smolin, and S. Speziale, J.Phys.A **A43**, 445401 (2010), 1004.4866.
 - [45] J. Hartle and S. Hawking, Phys.Rev. **D28**, 2960 (1983).
 - [46] R. Percacci, Phys.Lett. **B144**, 37 (1984).
 - [47] R. Percacci, PoS **ISFTG2009**, 011 (2009), 0910.5167.
 - [48] F. Nesti and R. Percacci, Phys.Rev. **D81**, 025010 (2010), 0909.4537.
 - [49] In the second-order formulation of gravity, the Christoffel connection is indeed a $GL(4)$ -valued connection but it is constructed from $g_{\mu\nu}$ and therefore not usually thought of as the fundamental variable.
 - [50] A similar remark applies to the Palatini first-order formu-

lation in which the dynamical variables are the $GL(4)$ -valued gauge field $\Gamma_{\mu\nu}^\rho$, the affine connection, and the metric $g_{\mu\nu}$, which again does not appear to have a natural counterpart in gauge theory.

- [51] An interesting alternative is in schemes that unify the $SO(1, 3)$ gauge field with gauge fields of grand unified

theories into a connection $\mathcal{A}_{\Gamma\mu}^\Omega$ for a larger Lie group \mathcal{G} [46–48]. In these theories there is also taken to exist an object θ_μ^Ω , which is a \mathcal{G} vector and space-time one-form. It is this object which may display a symmetry breaking solution to play the role of the $SO(1, 3)$ -valued co-tetrad.